Spurious Mode Generation in Nonuniform Waveguide*

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Summary—**This paper deals with the problem of a nonuniform waveguide joining two uniform ones and the spurious modes generated by it when a pure mode is incident in one of the uniform waveguides.**

The generalised telegraphist's equations are stated and transformed into a set of differential equations for the amplitudes of the forward and backward travelling waves. The expressions for the coupling coefficients between the various modes are given and analysed.

By making certain assumptions, the differential equations are solved and the amplitudes of the modes are given in a closed form.

Subject to these same assumptions, it is proved that the power in the spurious modes may be kept below any predetermined level, provided the nonuniform waveguide is sufficiently gradual.

I. Introduction

WAVEGUIDES whose cross sections change along the axis have been frequently investigated. A general solution was first given by Stevenson,[1] who expanded the field intensities into a series of the cross sectional wave functions. Using the same approach generalised telegraphist's equations were derived independently at the same time by Schelkunoff,[2] Reiter,[3] and Katzenelenbaum.[4] Since then, these methods were used successfully for solving a large number of problems.[5–13]

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[1] A. F. Stevenson, "General theory of electromagnetic horns," *J. Appl. Phys.*, vol. 22, pp. 1447–1454; December, 1951.

[2] S. A. Schelkunoff, "Conversion of Maxwell's equations into generalised telegraphist's equations," *Bell Sys. Tech. J.*, vol. 34, pp. 995–1044; September, 1955.

[3] G. Reiter, "Connection of two waveguides by a waveguide of variable cross-section," (In Hungarian) thesis from applied mathematics, University Eotvos Lorand, Budapest; June, 1955.

[4] B. Z. Katzenelenbaum, "Nonuniform waveguides with slowly changing parameters," (In Russian) *Dokl. Akad. Nauk, USSR*, vol. 12, pp. 711–714; 1955.

[5] B. Z. Katzenelenbaum, "Long symmetrical waveguide taper for H_{01} wave," (In Russian) *Radiotek. Elek.*, vol. 2, pp. 531–538; May, 1957.

[6] S. P. Morgan, "Theory of curved circular waveguide containing an inhomogeneous dielectric," *Bell Sys. Tech. J.*, vol. 36, pp. 1209–1252; September, 1957.

[7] Y. Shimizu, "Theory of transmitting circular electric wave around bends," Congres Internatl. Circuits et Antennes Hyperfrequences, Paris, France; October, 1957.

[8] B. Oguchi and M. Kato, "The effects of circular TE_{1m} waves on the propagation of circular TE_{01} wave in a curved waveguide," Congres Internatl. Circuits et Antennes Hyperfrequences, Paris, France; October, 1957.

[9] H. G. Unger, "Circular electric wave transmission through serpentine bends," *Bell Sys. Tech. J.*, vol. 36, pp. 1279–1291; September, 1957.

[10] H. G. Unger, "Normal mode bends for circular electric waves," *Bell Sys. Tech. J.*, vol. 36, pp. 1292–1307; September, 1957.

[11] B. Z. Katzenelenbaum, "On the general theory of nonuniform waveguides," (In Russian) *Dokl. Akad. Nauk, USSR*, vol. 116, pp. 203–206; 1957.

[12] B. Z. Katzenelenbaum, "On the theory of nonuniform waveguides with slowly changing parameters," Congres Internatl. Circuits et Antennes Hyperfrequences, Paris, France; October, 1957.

[13] H. G. Unger, "Circular waveguide taper of improved design," *Bell Sys. Tech. J.*, vol. 37, p. 899; July, 1958.

In the present paper, the amplitudes of the spurious modes, due to a nonuniform waveguide section, are determined. In Section II, Reiter's equations are transformed into a differential equation system in terms of forward and backward travelling waves. In Section III, the coefficients of this differential equation system are computed, and a few general conclusions are drawn. The differential equation system is simplified and solved by using a few approximations in Section IV, while in Section V, a theorem on sufficiently gradual nonuniform waveguides is proved. Section VI deals with two practical examples illustrating the application of the formulae derived.

II. The Generalised Telegraphist's Equations

Let the uniform waveguide G_1 extend from $z=-\infty$ to $z=0$, and the uniform waveguide G_2 from $z=L$ to $z=\infty$ (Fig. 1.). Let us connect them by a nonuniform waveguide which has the following properties: the equation of the surface is $F(x, y, z)=0$, which is differentiable as a function of z. A plane perpendicular to the z axis cuts this surface in a single, closed curve; the cross-section of the nonuniform waveguide. The interior of any cross section is denoted by $S(z)$, and its boundary by $C(z)$. The cross-sections at $z=0$ and $z=L$ correspond to those of the uniform waveguides G_1 and G_2 respectively.

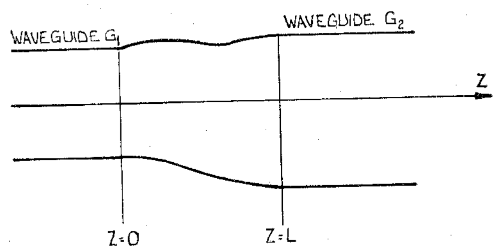

Fig. 1—A nonuniform waveguide section.

Our purpose is to determine the spurious modes in waveguide G_2, if waveguide G_1 is fed by a pure mode (subsequently called the main mode).

The nonuniform waveguide may be regarded as a system of coupled transmission lines, where the coupling coefficients are functions of z. The field intensities in the nonuniform waveguide may be represented by equivalent voltages and currents. The differential equation system for these voltages and currents is known as the generalised telegraphist's equation. Neglecting losses, we can use the following form as derived by Reiter:[14]

[14] G. Reiter, "Generalised telegraphist's equation for waveguides of varying cross-section," presented at the Convention on Long Distance Transmission by Waveguide," London, Eng., 1959; January 29–30.

$$-\frac{dV_i}{dz} = j\beta_i K_i I_i - \sum_p T_{pi} V_p$$
$$-\frac{dI_i}{dz} = j\frac{\beta_i}{K_i} V_i + \sum_p T_{ip} I_p \tag{1}$$

where i and p denote arbitrary modes (for the time being there is no need to discriminate the E and H modes). V_i and I_i are the equivalent voltages and currents for the mode i. β_i is the propagation coefficient, and K_i is the wave impedance. T_{pi} and T_{ip} represent the voltage and current transfer coefficients.

We point out that there is no mutual impedance between the voltages and currents of *different* modes, and there is a simple connection between the current and voltage transfer coefficients.

The transfer coefficient T_{pi} may be expressed as follows;

$$T_{pi} = \int_{S(z)} \bar{e}_p \frac{\partial \bar{e}_i}{\partial z} dS \tag{2}$$

where $\bar{e}_p$ and $\bar{e}_i$ are the mode vector functions[15] of the corresponding modes, which satisfy the normalisation conditions,

$$\int_{S(z)} |\bar{e}_p|^2 dS = 1. \tag{3}$$

For writing the mode vector functions, we must differentiate between E and H modes. For E modes, (subscripts in parentheses)

$$\bar{\mathbf{e}}_{(p)} = -\nabla_t \psi_{(p)}. \tag{4}$$

For H modes, (subscripts in brackets)

$$\bar{\mathbf{e}}_{[p]} = \bar{\mathbf{z}}_0 x \nabla_t \psi_{[p]} \tag{5}$$

where

∇_t = the gradient operator transverse to the z axis

z_0 = the unit vector in the direction of the z axis.

The $\psi_{(p)}$ and $\psi_{[p]}$ functions satisfy the differential equations

$$\nabla_t^2 \psi_{(p)} + h_{(p)}^2 \psi_{(p)} = 0$$
$$\psi_{(p)} = 0 \text{ on } C(z) \tag{6}$$

and

$$\nabla_t^2 \psi_{[p]} + h_{[p]}^2 \psi_{[p]} = 0$$
$$\frac{\partial \psi_{[p]}}{\partial n} = 0 \text{ on } C(z) \tag{7}$$

where

$$h_{[p]} = (k^2 - \beta_{[p]}^2)^{1/2}$$
$$h_{(p)} = (k^2 - \beta_{(p)}^2)^{1/2}$$
$$k = 2\pi/\lambda.$$

[15] N. Marcuritz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., p. 4; 1951.

For the description of a wave phenomenon, the representation in terms of forward and backward travelling waves seems to be more suitable. Therefore, assuming that

$$K_i \neq 0 \quad \text{and} \quad K_i \neq \infty, \tag{7a}$$

we introduce as new variables the amplitudes of the forward and backward travelling waves, A_i^+ and A_i^-, by the relations

$$V_i = K_i^{1/2}(A_i^+ + A_i^-)$$
$$I_i = K_i^{-1/2}(A_i^- + A_i^-). \tag{8}$$

Substituting (8) into (1) we obtain the following differential equations for coupled travelling waves;

$$\frac{dA_i^+}{dz} = -j\beta_i A_i^+ - \frac{1}{2}\frac{d(\ln K_i)}{dz} A_i^- + \sum_p (S_{ip}^+ A_p^+ + S_{ip}^- A_p^-)$$
$$\frac{dA_i^-}{dz} = -\frac{1}{2}\frac{d(\ln K_i)}{dz} A_i^+ + j\beta_i A_i^- + \sum_p (S_{ip}^- A_p^+ + S_{ip}^+ A_p^-) \tag{9}$$

where S_{ip}^+ is the forward and S_{ip}^- the backward coupling coefficient. Both may be expressed in terms of the transfer coefficients as follows:

$$S_{ip}^{\pm} = \frac{1}{2}\left[\frac{K_p^{1/2}}{K_i^{1/2}} T_{pi} \mp \frac{K_i^{1/2}}{K_p^{1/2}} T_{ip}\right]. \tag{10}$$

If the waveguide G_1 is fed by a mode m, the boundary conditions for the differential equation system are as follows;

$$A_m^+(0) = A_0, \qquad A_m^-(L) = 0$$
$$A_i^+(0) = 0 \qquad A_i^-(L) = 0 \qquad (i \neq m). \tag{11}$$

III. The Transfer and Coupling Coefficients

Substituting the mode vector functions into (2), using Green's and Stokes' theorems, the relations

$$K_{[i]} = \frac{\omega\mu}{\beta_{(i)}}, \qquad K_{(i)} = \frac{\beta_{(i)}}{\omega\epsilon}, \qquad k^2 = \omega^2\mu\epsilon \tag{12}$$

and the identities

$$\frac{\partial \psi_{(i)}}{\partial z} \equiv -\frac{\partial \psi_{(i)}}{\partial n} \tan\theta;$$
$$\frac{\partial}{\partial z}\frac{\partial \psi_{[i]}}{\partial n} \equiv -\frac{\partial^2 \psi_{[i]}}{\partial n^2} \tan\theta \text{ on } C(z), \tag{13}$$

the transfer and coupling coefficients may be expressed by line integrals as follows;[16]

[16] Similar expressions can be derived for the cases when $h_{(i)} = h_{(p)}$ and $h_{[i]} = h_{[p]}$.

$$T_{(i)(p)} = \frac{h_{(p)}^2}{h_{(i)}^2 - h_{(p)}^2} \oint_{C(z)} \tan\theta \frac{\partial \psi_{(i)}}{\partial n} \frac{\partial \psi_{(p)}}{\partial n} ds; \qquad h_{(i)} \neq h_{(p)} \tag{14}$$

$$T_{(i)[p]} = 0, \tag{15}$$

$$T_{[i](p)} = -\oint_{C(z)} \tan\theta \frac{\partial \psi_{[i]}}{\partial s} \frac{\partial \psi_{(p)}}{\partial n} ds \tag{16}$$

$$T_{[i][p]} = \frac{h_{[i]}^2}{h_{[p]}^2 - h_{[i]}^2} \oint_{C(z)} \tan\theta \psi_{[i]} \frac{\partial^2 \psi_{[p]}}{\partial n^2} ds; \qquad h_{[i]} \neq h_{[p]} \tag{17}$$

$$T_{(i)(i)} = S^-_{(i)(i)} = -\frac{1}{2} \oint_{C(z)} \tan\theta \left(\frac{\partial \psi_{(i)}}{\partial n} \right)^2 ds \tag{18}$$

$$T_{[i][i]} = S^-_{[i][i]} = -\frac{1}{2} \oint_{C(z)} \tan\theta \left(\frac{\partial \psi_{[i]}}{\partial s} \right)^2 ds \tag{19}$$

$$S^+_{(i)(i)} = S^+_{[i][i]} = 0 \tag{20}$$

$$S^{\pm}_{(i)(p)} = \frac{\beta_{(i)} h_{(p)}^2 \pm \beta_{(p)} h_{(i)}^2}{2\sqrt{\beta_{(i)}\beta_{(p)}}\,(h_{(i)}^2 - h_{(p)}^2)} \oint_{C(z)} \tan\theta \frac{\partial \psi_{(i)}}{\partial n} \frac{\partial \psi_{(p)}}{\partial n} ds; \qquad h_{(i)} \neq h_{(p)} \tag{21}$$

$$S^{\pm}_{(i)[p]} = \frac{k}{2\sqrt{\beta_{(i)}\beta_{[p]}}} \oint_{C(z)} \tan\theta \frac{\partial \psi_{(i)}}{\partial n} \frac{\partial \psi_{[p]}}{\partial s} ds \tag{22}$$

$$S^{\pm}_{[i][p]} = \frac{\beta_{[i]} h_{[p]}^2 \oint_{C(z)} \tan\theta \psi_{[p]} \frac{\partial^2 \psi_{[i]}}{\partial n^2} ds \pm \beta_{[p]} h_{[i]}^2 \oint_{C(z)} \tan\theta \psi_{[i]} \frac{\partial^2 \psi_{[p]}}{\partial n^2} ds}{2\sqrt{\beta_{[i]}\beta_{[p]}}\,(h_{[i]}^2 - h_{[p]}^2)} \tag{23}$$

where

ω = angular frequency
μ = permeability
ϵ = dielectric constant
θ = angle between the outward normal to $C(z)$ and the normal to the nonuniform waveguide (Fig. 2)
ds = an element of the $C(z)$ curve.

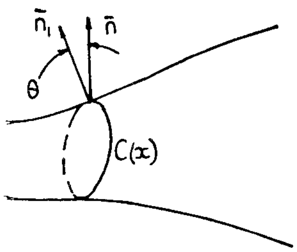

Fig. 2—Parameters used for the description of the nonuniform waveguide. $\bar{n}$—the outward normal to the boundary curve, $\bar{n}_1$—the outward normal to the nonuniform waveguide.

A few interesting conclusions may be drawn from the above Equations.

1) The transfer voltage coefficient from an E mode to an H mode is zero (15).

2) The transfer current coefficient from an H mode to an E mode is zero (15).

3) The S^+ matrix is skew-symmetric (10).

4) The S^- matrix is symmetric (10).

5) The backward coupling coefficient into the same mode is frequency independent (18), (19).

6) The coupling coefficients change very little with frequency, if it is not too near to the cut-off frequency (21)–(23).

7) The absolute values of the forward and backward coupling coefficients from an H mode into an E mode (or from an E mode into an H mode) are identical (22).

8) If on $C(z)$ either $\tan\theta$, or $\partial\psi_{[p]}/\partial s$ equal zero, there is no coupling between the H_p mode and any of the E modes. (For example no E modes are generated by changing the dimensions of the broad side of a rectangular waveguide, which supports an H_{0n} mode) (22).

9) The coupling of any one H mode to another (or any E mode to another) is larger the nearer their corresponding cut-off numbers are. This is only a rough rule, because the effect of the line integrals in (21) and (23) is not taken into account.

IV. An Approximate Solution of the Differential Equation System

We shall consider only such nonuniform waveguides in which the expected power conversion into the spurious modes is small by comparison with the power in the main mode. For gradual transitions, this is generally true. An exception arises however when a spurious mode has the same cut-off number as the main mode at each cross section. Excluding the latter case, we may assume that neither the spurious modes nor the reflection in the main mode have any effect on the forward propagation of the main mode. (This assumption was frequently used in nonuniform transmission line theory[17] and led to the linearisation of the Riccatti differential equation.)

Thus we have to solve for $A_m^+(z)$ the following simple differential equation;

$$\frac{dA_m^+}{dz} = -j\beta_m A_m^+, \qquad A_m^+(0) = A_0, \tag{24}$$

whence

$$A_m^+(z) = A_0 \exp\left[-j\int_0^z \beta_m dz\right]. \tag{25}$$

We assume further that the backward traveling main mode and both the forward and backward travelling spurious modes are excited only by the main mode; *i.e.*, we neglect the interaction of the different spurious modes among themselves, and the interaction between a forward and backward travelling spurious mode. Because of the restriction (7a) the solution will be valid only for those modes which are above cut-off *everywhere* in the nonuniform waveguide.

Subject to the above approximations, the following differential equations can be written

$$\begin{aligned} &\frac{dA_m^-}{dz} - j\beta_m A_m^- = \left(S_{mm}^- - \frac{1}{2}\frac{d(\ln K_m)}{dz}\right)A_m^+ \\ &\frac{dA_i^+}{dz} + j\beta_i A_i^+ = S_{im}^+ A_m^+ \\ &\frac{dA_i^-}{dz} - j\beta_i A_i^- = S_{im}^- A_m^+. \end{aligned} \tag{26}$$

Taking into account the boundary conditions (11), the solutions of the differential equations are as follows;

$$A_m^-(0) = -\exp\left[-j\int_0^L \beta_m dz\right]\int_0^L \left(S_{mm}^- - \frac{1}{2}\frac{d(\ln K_m)}{dz}\right)\exp\left[-j2\int_0^z \beta_m dz\right]dz \tag{27}$$

$$\left.\begin{matrix} A_i^+(L) \\ A_i^-(0) \end{matrix}\right\} = \pm \exp\left[-j\int_0^L \beta_i dz\right]\int_0^L S_{im}^\pm \exp\left[-j\int_0^z (\beta_m \mp \beta_i)dz\right]dz. \tag{28}$$

The amplitudes of the forward and backward converted spurious modes have been computed recently by Katzenelenbaum,[11,12] using a different approach. The equations given in this paper are believed to be more accurate because no approximations were used in the determination of the coupling coefficients.

V. A Theorem on Sufficiently Gradual Tapers

If the equation of the surface of the nonuniform waveguide, $F(x, y, z)=0$, is given, the functions $S_{im}^\pm(z)$, $\beta_i(z)$, $\beta_m(z)$ may be determined. Then subject to the validity of the approximations used in the previous Section, the amplitudes of the forward and backward travelling spurious mode i may be obtained from (28). Let us investigate how these amplitudes will change if the nonuniform waveguide is lengthened by a factor σ while its other dimensions are retained. The equation of the new nonuniform wave guide is

$$F\left(x, y, \frac{z}{\sigma}\right) = 0, \tag{29}$$

the coupling coefficient is

$$\frac{1}{\sigma} S_i^{m\pm}\left(\frac{z}{\sigma}\right), \tag{30}$$

the propagation coefficients are

$$\beta_i\left(\frac{z}{\sigma}\right) \text{ and } \beta_m\left(\frac{z}{\sigma}\right), \tag{31}$$

and the amplitude of the spurious mode i is given as follows;

$$\left.\begin{matrix} |A_i^+(\sigma L)| \\ |A_i^-(0)| \end{matrix}\right\} = \left|\int_0^L S_{im}^\pm(t) \cdot \exp\left\{-j\sigma\int_0^t [\beta_m(t) \mp \beta_i(t)]dt\right\}dt\right|, \tag{32}$$

where $t=z/\sigma$ and L is the length of the original nonuniform waveguide.

Using a transformed form of the Riemann lemma, it may be easily shown that the amplitude of any spurious mode may be held below any predetermined level by choosing σ sufficiently large.[18] Hence, if a pure mode is incident at the input, the mode at the output may be made arbitrarily pure. Thus a sufficiently gradual nonuniform waveguide may be represented by a single nonuniform transmission line.

[17] F. Bolinder, "Fourier transforms in the theory of inhomogeneous transmission lines," *Trans. Roy. Inst. Tech.*, Stockholm, Sweden, no. 48, p. 84; 1951.

[18] This does not mean that by making the nonuniform waveguide longer it will be necessarily better.

Examples

For the first example, let us take a rectangular waveguide, with b its narrow wall a function of z, and compute the reflection of the H_{01} mode. Then

$$\frac{d(\ln K_{[01]})}{dz} = 0 \tag{33}$$

$$S^{-}_{[01][01]} = -\frac{1}{2}\frac{d(\ln b)}{dz} \tag{34}$$

where

$$A^{-}_{[01]} = \exp j\beta_{[01]}L \int_0^L \frac{d(\ln b)}{dz} \exp\left[-j2\beta_{[01]}z\right]dz. \tag{35}$$

Eq. (35) agrees with the result derived by ordinary transmission line theory, but an essential difference is implied.

By classical transmission line theory, the *only* reason for reflection is a change in impedance, and for that reason it has been assumed that the characteristic impedance of the rectangular waveguide (excited in the dominant mode) is proportional to its height. This assumption certainly led to a correct result, but it had to introduce the concept of characteristic impedance, which has a very limited scope in dealing with waveguides.

However, approaching the rectangular waveguide from the viewpoint of general nonuniform waveguide theory, it is obvious from (27) that even in first order approximation—reflections are due to *two* reasons: 1) change in wave impedance, and 2) backward coupling into the same mode.

Thus—in our interpretation—the reflection in the above rectangular waveguide is due to a backward coupling and not a change in impedance.

Let us take for the second example a circularly symmetrical taper, which connects two circular waveguides of different diameters. We shall compute the amplitude of the spurious modes H_{MN}, E_{MN} in waveguides G_1 and G_2, when waveguide G_1 is fed by an H_{mn} mode. The calculation applies only to modes above cut-off at every cross-section of the taper.

Substituting the necessary expressions into (28), we get the amplitudes of the spurious modes in the following form;

$$\left.\begin{matrix} A^{+}_{[MN]}(L) \\ A^{-}_{[MN]}(0) \end{matrix}\right\} = 0 \qquad \text{if } M \neq m. \tag{36}$$

$$\left.\begin{matrix} A^{+}_{[mN]}(L) \\ A^{-}_{[mN]}(0) \end{matrix}\right\} = \pm \exp\left[\mp j\int_0^L \beta_{[mN]}dz\right]\int_0^L S^{\pm}_{[mN][mn]} \cdot \exp\left[-j\int_0^z (\beta_{[mn]} \mp \beta_{[mN]})dz\right)dz; \quad n \neq N \tag{37}$$

where

$$S^{\pm}_{[mN][mn]} = \frac{\beta_{[mN]}\chi^2_{[mn]}(\chi^2_{[mN]} - m^2) \pm \beta_{[mn]}\chi^2_{[mN]}(\chi^2_{[mn]} - m^2)}{\beta_{[mN]}\beta_{[mn]}(\chi^2_{[mN]} - \chi^2_{[mn]})} \frac{d(\ln a)}{dz}. \tag{38}$$

$\chi_{[mn]}$ — the nth root of the Bessel function $j_M'(x)$,
a — the radius of the cross-section at z.

$$\left.\begin{matrix} A^{+}_{(MN)}(L) \\ A^{-}_{(MN)}(0) \end{matrix}\right\} = 0 \qquad \text{if } M \neq m \tag{39}$$

$$\left.\begin{matrix} A^{+}_{(mN)}(L) \\ A^{-}_{(mN)}(0) \end{matrix}\right\} = \pm \exp\left[\mp j\int_0^L \beta_{(mN)}dz\right]\int_0^L S^{\pm}_{(mN)[mn]} \cdot \exp\left[-j\int_0^z (\beta_{[mn]} \mp \beta_{(mN)})dz\right]dz \tag{40}$$

where

$$S^{\pm}_{(mN)[mn]} = \frac{k}{\sqrt{\beta_{[mn]}\beta_{(mH)}}} \frac{m}{\sqrt{\chi_{[mn]} - m^2}} \frac{d(\ln a)}{dz}. \tag{41}$$

The amplitudes of the forward and backward H_{mn} and E_{mn} modes are given in terms of integrals which can be evaluated by any one of the approximate integration methods when the shape of the taper is given.

VI. Conclusions

A differential equation system in terms of forward and backward traveling waves has been obtained for a general nonunform waveguide. The forward and backward coupling coefficients between the different modes have been given and some general conclusions have been drawn about their properties. Assuming that all the investigated modes are above cut-off at every cross-section of the nonuniform waveguide, and the change in axial direction is gradual, the differential equation system has been solved. The amplitudes of the spurious modes at the beginning and at the end of the nonuniform waveguide have been given in a closed form.

It has been proved that by making the nonuniform waveguide more and more gradual, the amplitudes of all the spurious modes tend to zero. Thus—assuming a pure incident mode—a sufficiently gradual nonuniform waveguide may be represented by a single nonuniform transmission line.

VII. Acknowledgment

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